Modeling of the simple shear deformation of sand: effects of principal stress rotation

Marte Gutierrez · J. Wang · M. Yoshimine

Abstract The paper presents a simple constitutive model for the behavior of sands during monotonic simple shear loading. The model is developed specifically to account for the effects of principal stress rotation on the simple shear response of sands. The main feature of the model is the incorporation of two important effects of principal stress on stress–strain response: anisotropy and non-coaxiality. In particular, an anisotropic failure criterion, cross-anisotropic elasticity, and a plastic flow rule and a stress–dilatancy relationship that incorporate the effects of non-coaxiality are adopted in the model. Simulations of published experimental results from direct simple shear and hollow cylindrical torsional simple shear tests on sands show the satisfactory performance of the model. It is envisioned that the model can be valuable in modeling in situ simple shear response of sands and in interpreting simple shear test results.

Keywords Constitutive model · Dilatancy · Granular materials · Non-coaxiality · Simple shear

1 Introduction

One of the most common types of in situ soil deformation is that of simple shear. Simple shear condition is predominant during earthquake shaking of level grounds when deformation propagates vertically upwards from the bedrock in the form of shear waves. Soils within localized failure zones also deform in simple shear. Simple shear deformation can be simulated in element tests using the direct simple shear (DSS) device, or the hollow cylindrical torsional (HCT) simple shear device. The main drawback of simple shear devices is the non-uniformity of the stress and strain distributions within the soil specimen. Despite this shortcoming, simple shear testing continues to have wide use in practice due to the relevance of the test to in situ conditions and the ease of carrying out the test.

Another drawback in the use of simple shear testing is the difficulty of interpreting the test results. A key feature of simple shear loading is that the principal stress directions are not fixed but rotate during shearing. As a result, the orientations of the failure planes are not a priori known and depend on the degree of principal stress rotation. Simple shear devices, where the sample is constrained from deforming laterally, have only two degrees of freedom corresponding to the shear and vertical deformations. Thus, the principal stress rotation cannot be directly controlled and only limited rotation can be achieved [22]. Experimental determination of the degree of principal stress rotation requires measurement of the variation of the lateral confining stress during shearing. However, this measurement is difficult to perform and is not usually carried out in routine DSS tests.

Due to anisotropy, the response of sands during loading depends on the principal direction. Experimental data show that the shear strength and deformation of sands are
dependent on the loading direction. Due to anisotropy, soils are expected to deform during principal stress rotation even when the shear stress level or the mobilized friction angle is kept constant [1, 8, 13].

Another important effect of principal stress rotation is the non-coincidence or non-coaxiality of the principal stress and principal plastic strain increment directions. Extensive experimental data from Hollow Cylindrical Testing have shown that the principal stress direction lags behind the principal plastic strain increment direction when the principal stresses are rotated [1, 7, 8, 13]. The deviation or non-coaxiality between the principal stress and principal plastic strain increment directions is significant for loading involving pure stress rotations at low mobilized friction angles. Non-coaxiality has important implications on the post-localization response of granular soils as shown by [8, 25].

Early attempts at modeling simple shear behavior of soils were made by [6, 14, 15, 18, 20]. These models did not fully account for the effects of principal stress rotation. Recently, models that consider the effects of principal stress rotation on simple shear response have been proposed, e.g. [19, 26]. However, most of these models are based on complicated theories such as hypo-plasticity, making them difficult to use in routine geotechnical practice.

The objective of this paper is to present a simple constitutive model for the behavior of sands during monotonic simple shear loading that fully accounts for the effects of principal stress rotation. The main feature of the model is the incorporation of the two important effects of principal stress on stress–strain response: anisotropy and non-coaxiality. In particular, an anisotropic failure criterion, cross-anisotropic elasticity, and a plastic flow rule and a stress–dilatancy relationship that incorporate the effects of principal stress rotation are adopted in the model. The model uses simple equations, and the implementation does not require complicated numerical integration. The model can be easily and conveniently implemented in a spreadsheet environment, allowing for their ease of use in practice. The model is validated against published experimental DSS and HCT data on sands.

2 Model for non-coaxial simple shear deformation of sand

The analytical model for the non-coaxial simple shear deformation of sand is formulated in two-dimensional plane strain conditions where the stress and strain increment tensors are defined as

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix}, \quad d\varepsilon_{ij} = \begin{bmatrix} d\varepsilon_x & d\varepsilon_{xy} \\ d\varepsilon_{xy} & d\varepsilon_y \end{bmatrix}$$  \hspace{1cm} (1)

Note that $d\varepsilon_{xy} = d\gamma_{xy}/2$, where $d\gamma_{xy}$ is the engineering shear strain increment. The two-dimensional plane strain assumption implies that the out-of-plane strains are zero, i.e., $d\varepsilon_{zz} = d\varepsilon_{xz} = \ldots = 0$. All stresses are assumed to be effective, and will be denoted without the symbol prime (').

In formulating the model, the stress invariants $s$ and $t$ (which correspond to the mean and shear stresses, respectively, in the plane of deformation) and strain rate invariants $dv$ and $d\gamma$ (which correspond to volumetric and shear strain rates, respectively) will be used. These invariants are defined as follows:

$$s = \frac{1}{2}(\sigma_x + \sigma_y), \quad t = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \sigma_{xy}^2)^{1/2}$$  \hspace{1cm} (2)

$$dv = d\varepsilon_x + d\varepsilon_y, \quad d\gamma = ((d\varepsilon_x - d\varepsilon_y)^2 + 4d\varepsilon_{xy}^2)^{1/2}$$  \hspace{1cm} (3)

From the Mohr circle for stress (Fig. 1a), it can be seen that $s$ is equal to the distance to the center of circle from the origin, and $t$ is equal to the radius of the circle. Similarly, from the Mohr circle for strain (Fig. 1b), $dv/2$ is equal to the distance to the center of circle from the origin, and $d\gamma/2$ is equal to the radius of the circle.

![Fig. 1 Mohr’s circles: a stress, b incremental strain for simple shear conditions](image)

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2.1 Non-coaxial stress–dilatancy relationship

Coaxiality is a fundamental assumption in continuum mechanics. It implies that the directions of principal plastic strain increment and principal stresses coincide. Gutierrez and Ishihara [7] showed that coaxiality does not hold for loadings involving principal stress rotation, and at the same time, conventional plasticity models expressed in terms of the usual stress and plastic strain increment invariants implicitly assume coaxiality in the plastic flow rule. They showed that one way to integrate the effects of non-coaxiality in the plastic deformation of granular soils is by modifying the expressions for energy dissipation and stress–dilatancy relations. They derived the following stress–dilatancy relationship that is valid for loadings involving varying degrees of principal stress rotation by accounting for the degree of non-coaxiality in the calculation of dissipated energy:

\[
\frac{d\sigma_d}{d\gamma} = \sin \phi_e - c_t \frac{t}{s}
\]  

(4)

where \(d\sigma_d\) is the volumetric strain increment due to dilatancy, and \(\phi_e\) is the friction angle at phase transformation, and \(c\) is the non-coaxiality factor equal to:

\[c = \cos 2\Delta\]

(5)

The non-coaxiality angle \(\Delta\) is equal to the deviation between the principal stress and the principal strain increment directions:

\[\Delta = |\alpha - \beta|\]

(6)

The angles \(\alpha\) and \(\beta\) are shown in Fig. 1 and are defined from the Mohr’s circle of stress and strain increment as

\[\tan 2\alpha = \frac{2\sigma_{xy}}{\sigma_y - \sigma_x}, \quad \tan 2\beta = \frac{2d\sigma_{xy}}{d\sigma_y - d\sigma_x}\]

(7)

In terms of the dilation angle \(\psi\) and the mobilized friction angle \(\psi\), Eq. 4 can also be written as

\[\sin \psi = \sin \phi_e - c \sin \phi\]

(8)

\[\sin \psi = \frac{d\sigma_d}{d\gamma}, \quad \sin \phi = \frac{t}{s}\]

(9)

In addition to the above stress–dilatancy relation derived from Critical State Soil Mechanics, another widely used stress–dilatancy relation is the one derived by [23]. Gutierrez and Wang [9] recently extended Rowe’s stress–dilatancy relationship to account for non-coaxiality. In terms \(\psi\) and \(\phi\), the non-coaxial Rowe’s stress–dilatancy equation is given as

\[\sin \psi = \frac{\sin \phi_e - \sin \phi}{c(1 - \sin \phi \sin \phi_e)}\]

(10)

Eq. 10 was found to provide a better fit of experimental simple shear data than Eq. 8, and thus, Eq. 10 will be used below in the development of the simple shear model.

In simple shear, the lateral strain is equal to zero, \(d\varepsilon_l = 0\). As a result, the volumetric strain increment is equal to the vertical strain increment due to dilatancy, i.e., \(d\sigma_d = d\varepsilon_d\), and the dilatancy ratio \(d\varepsilon_d/d\gamma\) is equal to

\[\sin \psi = \frac{d\varepsilon_d}{\sqrt{d\varepsilon_d^2 + d\gamma^2}}\]

(11)

From the above equation, the simple shear dilatancy ratio \(d\varepsilon_d/d\gamma\) can be written as

\[\frac{d\varepsilon_d}{d\gamma} = \frac{\sin \psi}{\sqrt{1 - \sin^2 \psi}} = \tan \psi\]

(12)

2.2 Total vertical strain increment

In addition to the vertical strain increment from dilatancy \(d\varepsilon_d\), a change in the vertical stress will also induce a vertical strain \(d\varepsilon_{yc}\) due to compression of the soil sample. Since the soil sample is constrained from deforming laterally, the compression induced vertical strain increment \(d\varepsilon_{yc}\) is related to the vertical stress increment \(d\sigma_y\) by the constrained modulus \(M\):

\[d\varepsilon_{yc} = \frac{d\sigma_y}{M}\]

(13)

The total vertical strain increment \(d\varepsilon\) is now equal to

\[d\varepsilon = d\varepsilon_{yc} + d\varepsilon_d = \frac{d\sigma_y}{M} + (\tan \psi) d\gamma_{xy}\]

(14)

2.3 Shear stress–strain relationship

A relationship that is widely used in the modeling of the non-linear stress–strain response of soils is the hyperbolic stress–strain model. Here, the hyperbolic relation will be used in terms of the shear stress ratio \(\sigma_{xy}/\sigma_y\) and the shear strain \(\gamma_{xy}\):

\[\frac{\sigma_{xy}}{\sigma_y} = \frac{G_{\text{max}} \tan(\phi_{dss}) \gamma_{xy}}{\tan(\phi_{dss}) + G_{\text{max}} \gamma_{xy}}\]

(15)

where \(\phi_{dss}\) is the peak direct simple shear friction angle corresponding to the peak value of the direct simple shear stress ratio, i.e., \(\tan(\phi_{dss}) = (\sigma_{xy}/\sigma_y)\) and \(G_{\text{max}}\) is the maximum initial shear modulus. The above equation models a shear stress ratio \(\sigma_{xy}/\sigma_y\), increasing asymptotically towards the peak shear stress \(\phi_{dss}\) with increasing shear strain \(\gamma_{xy}\).

From the Mohr’s circle of stress (Fig. 1), the simple shear friction angle \(\phi_{dss} = \tan^{-1}(\sigma_{xy}/\sigma_y)\) can be related to the plane strain friction \(\phi = \sin^{-1}(\sigma_1 - \sigma_3)/(\sigma_1 + \sigma_3)\) and the principal stress direction \(\alpha\) as
\[
\tan \phi_{dss} = \frac{\sin \phi \sin 2\alpha}{1 + \sin \phi \cos 2\alpha}
\]  
(16)

Note that Eq. 16 is valid for all levels of the mobilized shear friction angle \(\phi\), including the peak condition, provided the corresponding \(\alpha\) value is used.

2.4 Principal stress rotation during simple shear loading

Oda and Konishi [18], Oda [16], and Ochiai [15] developed a simple expression for the amount of principal stress rotation that occurs during simple shear loading. They arrived at an expression which reads:

\[
\tan(\phi_{dss}) = \frac{\sigma_{xy}}{\sigma_y} = \kappa \tan \alpha
\]  
(17)

This equation implies a straight-line relationship between the simple shear stress ratio on a horizontal plane and the tangent of the angle that \(\sigma_1\) makes with the vertical axis. The constant \(\kappa\) can be shown to be related to the ratio of the initial horizontal to the initial vertical stress, which in most in situ cases is equal to the \(K_o\) value.

The above equation is compared with the simple shear experimental data of [4] in Fig. 2. As can be seen, Eq. 17 provides a good approximation of the principal stress rotation during simple shear loading of sands. The experimental results show some differences in the degree of principal stress rotation between the loose, medium dense and dense sand samples. However, the effects of density appear to be very small and will be neglected in the modeling. It is also worth noting from this figure that although a full 180° principal stress rotation cannot be achieved in simple shear loading, the magnitude of principal stress is still substantial, approaching as much as 60°.

2.5 Non-coaxiality angle in simple shear loading

The angle of non-coaxiality \(\Delta\) is evaluated using the plastic flow rule developed by [10] for plastic flow in the \((\sigma_y - \sigma_x)/2\) versus \(\sigma_{xy}\) stress rotation plane. This flow rule is described in Appendix, and gives the following expression for the non-coaxiality angle:

\[
\Delta = x - \xi - \frac{1}{2} \sin^{-1} \left( \frac{\sin \phi}{\sin \phi_p} \sin(2\alpha - 2\xi) \right)
\]  
(18)

where \(\phi\) is the mobilized friction angle, \(\phi_p\) is the peak friction angle, and \(\xi\) is the principal stress increment direction defined as

\[
\tan 2\xi = \frac{2d\sigma_{xy}}{d\sigma_y - d\sigma_x}.
\]  
(19)

2.6 Principal stress increment direction

To obtain an expression for the rotation of the principal stress increment direction \(\xi\), which is required in Eq. 18, it is only necessary to note that the stress increment vector should be tangential to the stress path. The stress path is given in Eq. 17 in terms of the stress ratio \(\sigma_{xy}/\sigma_y\) and the principal stress direction \(\alpha\) can be differentiated to obtain this tangent. First, Eq. 17 is re-written in terms of \(2\alpha\):

\[
\frac{\sigma_{xy}}{\sigma_y} = \kappa \left( \frac{\sqrt{1 + \tan^2 2\alpha} - 1}{\tan 2\alpha} \right)
\]  
(20)

Substituting Eq. 7 and solving for \(\sigma_x\):

\[
\sigma_x = \frac{\sigma_{xy}^2}{\kappa \sigma_y} + (1 - \kappa)\sigma_y
\]  
(21)

The differential form of the above equation can be written as

\[
d\sigma_x = \frac{2\sigma_{xy}d\sigma_{xy} - \sigma_{xy}^2}{\kappa \sigma_y} d\sigma_y + (1 - \kappa)d\sigma_y
\]  
(22)

It can be noted that when \(\sigma_{xy} = 0\), \(d\sigma_x = (1 - \kappa)d\sigma_y\), and as a result, it can be concluded that \(\kappa = (1 - K_o)\) when there is no shear stress applied.

The stress increment \(d\sigma_{xy}\) can be determined from \(d\gamma_{xy}\) and \(d\sigma_x\) using the incremental form of Eq. 15:

\[
d\sigma_{xy} = \frac{G_{mx} \tan \phi_{(dss)p} \gamma_{xy}}{\tan \phi_{(dss)p} + G_{mx} \gamma_{xy}} d\sigma_x + \frac{G_{mx} (\tan \phi_{(dss)p})^2}{(\tan \phi_{(dss)p} + G_{mx} \gamma_{xy})^2} d\gamma_{xy}
\]  
(23)
In case the change in effective vertical stress \(d\sigma_y\) is known or prescribed, this value together with \(d\sigma_z\) from Eq. 22 and \(d\sigma_x\) from Eq. 23 can be substituted in Eq. 19 to obtain the principal stress increment direction \(\xi\).

For drained direct simple shear test, a constant vertical stress, \(d\sigma_z = 0\), is usually used, and in this case,
\[
d\sigma_x = \frac{2\sigma_{xy}d\sigma_{xy}}{\kappa\sigma_y}, \quad d\sigma_y = 0
\]

(24)

The principal stress direction \(\xi\) (Eq. 19) can then be written as
\[
\tan 2\xi = \frac{-2d\sigma_y}{d\sigma_x}, \quad d\sigma_y = 0
\]

(25)

Combining Eqs. 22 and 24 gives:
\[
\tan 2\xi = -\kappa \frac{\sigma_y}{\sigma_{xy}} = -\kappa \frac{1}{\tan \phi_{dss}}, \quad d\sigma_y = 0
\]

(26)

For undrained loading condition, the change in the effective vertical stress \(d\sigma_z\) can be obtained assuming that the sample volume is constant during loading, and thus, \(d\sigma_z = 0\). Substituting this condition in Eq. 14 gives the vertical stress increment \(d\sigma_z\) in terms of the shear strain increment during undrained shearing:
\[
d\sigma_z = -M(\tan \psi)d\gamma_{xy}.
\]

(27)

2.7 Anisotropic friction angle and deformation moduli

Extensive experimental data show that the friction angle of sand is strongly dependent on the direction of the applied loading. Studies by [2, 13, 17, 24] showed that the friction angle of sand is highest when the major principal stress is oriented normal to the bedding plane, i.e., when \(\alpha = 0^\circ\), and decreases as \(\alpha\) is rotated from 0° to 90°. Typical variations of \(\phi\) as function of \(\alpha\) from experiments are shown in Fig. 3.

The effect of the major principal stress direction on the peak friction angle is represented in the model by the following relationship:
\[
\sin \phi_p(\alpha) = \frac{2k \sin \phi_p(0^\circ)}{(1 + k) - (1 - k) \cos(2\alpha)}
\]

(28)

where \(\phi_p(\alpha)\) is the direction-dependent friction angle, \(\phi_p(0^\circ)\) is the friction angle for \(\alpha = 0^\circ\), and \(k\) is the ratio of the sine of the friction angles for \(\alpha = 90^\circ\) and \(\alpha = 0^\circ\), i.e., \(k = \sin \phi_p(90^\circ)/\sin \phi_p(0^\circ)\). The validity of Eq. 28 can be seen in Fig. 3 where the equation is shown to adequately represent the trend in the anisotropy of the peak friction angle as manifested by the experimental data. The best fit of Eq. 28 through the experimental data gave a value of \(k = 0.84\), while the lower and upper bound curves gave values of 0.80 and 0.94, respectively. Combining Eqs. 16 and 28 gives the following variation in the peak simple shear friction angle \((\phi_{dss})_p\) with the major principal stress direction \(\alpha\):
\[
\tan(\phi_{dss})_p = \frac{2k \sin \phi_p(0^\circ) \sin 2\alpha}{(1 + k) - (1 - k \sin \phi_p(0^\circ)) \cos 2\alpha}
\]

(29)

In similar manner, experimental data show that the elastic deformation of sand is strongly dependent on the direction of loading. The small strain deformation of sands has been widely represented by cross-anisotropic elasticity where the deformation along the bedding plane is isotropic but different from the deformation perpendicular to the bedding plane [3, 5, 11, 12, 28]. For plane strain condition, cross-anisotropic elasticity requires four independent parameters, as shown in the following stress–strain relation:

\[
\begin{pmatrix}
\frac{d\sigma_x}{d\varepsilon_x} \\
\frac{d\sigma_y}{d\varepsilon_y} \\
\frac{d\sigma_z}{d\varepsilon_z}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{pmatrix}
\begin{pmatrix}
\frac{d\varepsilon_x}{d\sigma_x} \\
\frac{d\varepsilon_y}{d\sigma_y} \\
\frac{d\varepsilon_z}{d\sigma_z}
\end{pmatrix}
\]

(30)

where \(C_{11} = M_h = \) constrained modulus in the horizontal direction, \(C_{33} = M_v = \) constrained modulus in the vertical direction, and \(C_{12} = G_{sv} = \) shear modulus in the plane of anisotropy. Due to the zero lateral strain \((d\varepsilon_y = 0)\) condition for simple shear loading, only two of these parameters are required, and these are the vertical constrained modulus \(M_v = M\), and the shear modulus \(G_{sv} = G_{max}\). It can be shown that parameter \(C_{12}\) can be related to the \(K_o\) ratio by the relation:
\[
\frac{C_{12}}{M} = \frac{d\sigma_z}{d\sigma_y} = K_o
\]

(31)

To calculate \(d\sigma_z\), Eq. 31 is only used for consolidation, while Eq. 22 is used for shear loading.
3 Comparisons with experimental data

To illustrate its validity, the model described above is used to simulate the DSS test results obtained by [4] on Leighton-Buzzard sand, and the HCT results obtained by [27] on Kartal sand. Comparisons of model simulations and experimental results are shown in Figs. 4 and 5. The experimental data for Leighton-Buzzard sand are for drained tests, while the experimental data for Kartal sand are for undrained tests. Three sets of test results on Leighton-Buzzard sand with void ratios at the end of consolidation of \(e_o=0.739, 0.645\) and \(0.539\) are used in the validation of the model. These void ratios correspond, respectively, to loose, medium dense and dense relative densities. All tests were done at constant \(\sigma_y\) value of 100 kPa. For Kartal sand, the test results are for vertical consolidation stresses of \(\sigma_{vo}=40, 60\) and 80 kPa. The void ratios at the end of consolidation \(e_o\) for the three tests are very similar ranging from 0.870 to 0.878.

Table 1 lists the model parameters used in the simulations of the experimental results. Three of the parameters, namely the peak friction angle \(\phi_p\), the initial shear modulus \(G_{max}\), and the bulk modulus \(M\) are dependent on the initial void ratio \(e_o\) and the initial vertical stress \(\sigma_{vo}\). For Leighton-Buzzard sand, the principal stress rotation parameter \(\kappa\) appears to be independent of initial void ratio \(e_o\), while for Kartal sand \(\kappa\) varies with the initial vertical stress \(\sigma_{vo}\). The dependency of these parameters on \(e_o\) and \(\sigma_{vo}\) are given by the expressions in Table 1. The dependency of \(G_{max}\) on \(e_o\) and \(\sigma_{vo}\) follows the empirical relation by [21]. The other parameters, namely the friction angle at phase transformation \(\phi_c\) and anisotropy parameter \(k\) are relatively constant and independent of \(e_o\) and \(\sigma_{vo}\).

As can be seen in Figs. 4 and 5, the simulations of the simple shear response are very close to the experimental data. There is a good agreement between the stress–strain response, volumetric response (in drained tests), effective stress path (in undrained tests), and the directions of the

![Fig. 4 Comparison of experimental data and model predictions. Top row shear strain versus shear stress, middle row shear strain versus volumetric strain, and bottom row shear strain versus directions \(\alpha, \beta, \xi\) (experimental data from [4])](image)
For the Leighton-Buzzard, the predicted and measured principal strain increment direction \( b \) and the principal stress increment direction \( n \) are also in good agreement.

4 Conclusions

A key feature of the response of soils during simple shear loading is the rotation of the principal stresses. Early experimental studies by [15, 18, 22] showed that the principal stress rotation is limited (typically to less than 60°), and that it occurs mainly during small strain deformation. Consequently, the effects of principal stress rotation have been routinely neglected in the formulation of the simple shear response of soils. However, the recent emphasis on accurate representation of the pre-failure deformation of soils has made it essential to account for the effects of principal stress rotation. To meet this need, the paper presented a constitutive model that specifically

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**Table 1** Model parameters used for simulation of simple shear behavior of Leighton-Buzzard sand and Kartal sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Leighton-Buzzard sand</th>
<th>Kartal sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak friction angle ( \phi_a ) for ( \alpha = 0^\circ )</td>
<td>74.6( e_o ) – 7.7</td>
<td>82.1(( \sigma_v/p_a ) – 47.7</td>
</tr>
<tr>
<td>Friction angle at phase transformation ( \phi_c )</td>
<td>33.5</td>
<td>36.5</td>
</tr>
<tr>
<td>Shear modulus ( G_{\text{max}} )</td>
<td>90F(( e_o, \sigma_{yo} ))</td>
<td>110F(( e_o, \sigma_{yo} ))</td>
</tr>
<tr>
<td>Constrained modulus ( M )</td>
<td>–</td>
<td>902.7(( \sigma_v/p_a ) – 208.8</td>
</tr>
<tr>
<td>Stress rotation parameter ( \kappa )</td>
<td>0.55</td>
<td>2.95 – 2.73(( \sigma_v/p_a ))</td>
</tr>
<tr>
<td>Anisotropy parameter ( k )</td>
<td>0.85</td>
<td>0.85</td>
</tr>
</tbody>
</table>

\[
F(e_o, \sigma_{yo}) = p_a \sqrt{\sigma_{yo}/p_a(2.97 – e_o)^2/(1 + e_o)}, \text{ where } e_o = \text{initial void ratio, } \sigma_{yo} = \text{initial vertical stress, } p_a = \text{atmospheric pressure}
\]

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Fig. 5 Comparison of experimental data and model predictions. Top row shear strain versus shear stress, middle row effective stress path, and bottom row shear strain versus principal stress direction \( \alpha \) (experimental data from [27]).
accounts for the effects of principal stress rotation on the simple shear behavior of sands. The model incorporated two important effects of principal stress on stress–strain response, namely anisotropy and non-coaxiality. The model used an anisotropic failure criterion, cross-anisotropic elasticity, and a plastic flow rule and a stress–dilatancy relationship that incorporate the effects of non-coaxiality. The model was validated against published experimental results from direct simple shear and hollow cylindrical torsional simple shear tests on sands.

Appendix: Non-coaxiality angle due to principal stress rotation

The angle of non-coaxiality angle $\Delta$ required to determine the non-coaxiality parameter $c$ in the stress–dilatancy relationships given in Eqs. 9 and 10 is evaluated using the flow rule developed by [10] for plastic flow in the $(\sigma_y - \sigma_z)/2$ versus $\sigma_{xy}$ stress rotation plane. This flow rule is shown in Fig. 6. In this figure, the strain increment components $(d\varepsilon_y - d\varepsilon_z)$ and $2d\varepsilon_{xy}$ have been superimposed on the stress plane. Point A is the current stress point. Neglecting the effect of anisotropy, the failure surface is circular and centered at the origin of the $(\sigma_y - \sigma_z)/2$ versus $\sigma_{xy}$ axes. The distances $OA$ and $OB$ are equal to the current shear stress $t$ and the radius of the circular failure surface, respectively. These distances can be related to the current mean stress $s$, the mobilized friction angle $\phi$ and the peak friction angle $\phi_p$ as follows:

$$OA = s \sin \phi, \quad OB = s \sin \phi_p \quad (32)$$

On the $(\sigma_y - \sigma_z)/2$ versus $\sigma_{xy}$ stress plane, a stress vector makes an angle equal to $2\alpha$ from the $(\sigma_y - \sigma_z)/2$ axis (Eq. 7). Similarly, a stress increment vector makes an angle equal to $2\zeta$ from the $(\sigma_y - \sigma_z)/2$ axis (Eq. 19). On the $(d\varepsilon_y - d\varepsilon_z)$ versus $2d\varepsilon_{xy}$ strain increment plane, a strain increment vector has a length equal to the plastic shear strain increment $d\gamma$ and makes an angle equal to $2\beta$ from the $(d\varepsilon_y - d\varepsilon_z)$ axis. This plastic strain increment direction is evaluated as the normal to the failure surface at the conjugate point $A$ which is the intersection of the failure surface and the stress increment extended from the current stress point. This flow rule is based on the experimental observations that plastic flow on the $(\sigma_y - \sigma_z)/2$ versus $\sigma_{xy}$ stress plane is dependent on the stress increment direction [10]. This flow rule should be contrasted with conventional plasticity formulations where the plastic strain increment direction is evaluated at the current stress point independent of the stress increment direction. Details including the experimental verification of the flow rule are given in [10].

From triangle OAB in Fig. 6, the following angles can be obtained:

$$\angle OAB = \pi - (2\zeta - 2\alpha) \quad (33)$$

$$\angle BOA = 2\Delta \quad (34)$$

$$2\Delta = \pi - \angle OAB - \angle ABO \quad (35)$$

Using the law of sines:

$$\frac{OA}{\sin \angle ABO} = \frac{OB}{\sin \angle OAB} \quad (36)$$

$$\angle ABO = \sin^{-1} \left( \frac{OA}{OB} \sin \angle OAB \right) \quad (37)$$

Substituting Eqs. 32 and 33 in Eq. 37:

$$\angle ABO = \sin^{-1} \left( \frac{\sin \phi}{\sin \phi_p} \sin(\pi + 2\alpha - 2\zeta) \right) \quad (38)$$

Substituting Eqs. 33 and 38 in Eq. 34 gives:

$$2\Delta = 2\alpha - 2\zeta - \sin^{-1} \left( \frac{\sin \phi}{\sin \phi_p} \sin(\pi + 2\alpha - 2\zeta) \right) \quad (39)$$

Simplifying the above equation results in the expression for $\Delta$ given in Eq. 18.

References