A new probabilistic approach for predicting particle crushing in one-dimensional compression of granular soil

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Abstract

This paper presents a new probability-based method for the prediction of particle crushing in the one-dimensional compression of granular soil. The method is comprised of a joint-probability particle crushing criterion that takes into account the statistical particle-scale stress distribution information derived from the Discrete Element Method (DEM) simulations and Weibull’s distribution of particle crushing strength. A normalized tensile stress index $f$ and a diameter index $L_d$ are introduced to quantify the statistical particle-scale data. The method is further implemented in DEM simulations and verified by comparing the simulation results with the experimental data of oedometer tests of sands.

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1. Introduction

Particle crushing plays an important role in the mechanical properties of granular soils. A complete and thorough understanding of the macromechanical behavior of crushable sands and its underlying micromechanical principles is a prerequisite to the development of a new generation of physically rigorous constitutive models of sands. Despite the significant amount of research efforts devoted to this subject (e.g., Bolton, 1986; Cuccovillo and Coop, 1999), great difficulty still exists in the theoretical quantification and prediction of particle crushing, particularly when the influences of a number of particle-scale factors such as particle size/shape, fracture strength, soil grading, applied stress condition, etc., are to be accounted for.

By means of various types of laboratory tests, a rich body of knowledge has been gained over the years. For example, Luzzani and Coop (2002), Coop et al. (2004) and Bandini and Coop (2011), by using the breakage ratio ($B_r$) first proposed by Hardin (1985), revealed the relationship between particle breakage and the critical-state response of sands through a series of triaxial and ring shear tests. In their tests, the soil reached a stable grading at very large displacement, but the final grading was found to depend on the initial grading and the applied external stress. A series of oedometer tests conducted by Altuhafi and Coop (2011a, 2011b) clearly showed that a much larger amount of particle breakage occurred in poorly-graded samples than in well-graded samples and this trend continued to
a point where no significant particle breakage could be measured in very well-graded samples. By testing the well-graded silica sand, Nakata et al. (2001) found that bigger particles were more susceptible to asperity breaking and surface grinding, while smaller particles were more susceptible to major splitting. Due to the limitation of experimental techniques, what happens inside the sand and why this phenomenon occurs are not clear. Two opposite trends about the particle crushability within a loaded sand that need to be considered are as follows: large particles are safer because they tend to have more contacts than surrounding particles, whereas small particles are more difficult to break than large particles due to their higher average fracture strength. For well-graded samples, an internal dynamic balance might be established, leading to a stable state.

By assuming that the sand crushing produces a soil grading which obeys the fractal geometry, McDowell and Bolton (1998) derived a theoretical relationship between the 1D compression rate of a sand and its fractal geometry of grading. The most important assumption of the fractal theory is that it is the smallest particles that continue to fracture under the increasing macroscopic stress, and that the probability of fracture of the smallest particles is a constant. A conceptually different approach taken by Nakata et al. (1999) to predict the particle breakage within a triaxial sample was based on the statistical distribution of single particle fracture determined from single particle crushing tests, but an ad hoc assumption was made to estimate the particle-scale stress condition for the breakage prediction purpose. The assumptions adopted in the above studies, however, lack of supporting experimental evidence and are questionable because the particle-scale stress distribution within a loaded sand could be highly non-uniform. Einav (2007a, 2007b) developed a new continuum breakage mechanics theory for the constitutive modeling of brittle granules, using the relative breakage ratio as the damage internal variable; however, the micromechanics of particle breakage were not considered in his research.

In recent years, more sophisticated investigations into the particle breakage behavior within a loaded granular soil using the X-ray CT technique have emerged. The high-resolution CT images offered by laboratory nanofocus using the X-ray CT technique have emerged. The high-particle breakage behavior within a loaded granular soil however, the micromechanics of particle breakage were not relative breakage ratio as the damage internal variable; the constitutive modeling of brittle granules, using the development of a theoretically robust and numerically efficient particle breakage criterion.

As an alternative to the investigation of fundamental sand behavior, the Discrete Element Method first proposed by Cundall and Strack (1979) has made significant contributions towards unraveling the microscopic mechanisms of crushable soil behavior over the past two decades. In tackling the simulation of crushable soils, two common approaches exist in the literature: particle cluster approach and particle multiplication approach. The first approach bonds a group of elementary spheres together to form a porous agglomerate which can disintegrate to simulate particle breakage. It was used by a number of authors (e.g., Robertson, 2000; Cheng et al., 2003, 2004; Bolton et al., 2008; Wang and Yan, 2011, 2012) to investigate a wide range of fundamental soil behavior including soil yielding, plastic deformation, fractal crushing, strain localization and particle-scale energy allocation. The second approach, proposed by Lobo-Guerrero and Vallejo (2005a, 2005b), simulates particle breakage by replacing the original sphere by several smaller spheres. It was much less popular than the first approach, mainly because of the difficulty of establishing a physically sound breakage criterion for the prediction of particle breakage. However, the potential of this approach will be exploited in the current study through the construction of a theoretically robust and numerically efficient particle breakage criterion.

Another innovative and relevant method to predict particle breakage recently proposed by Marketos and Bolton (2007, 2009) was based on the statistical distribution of inter-particle contact forces derived from DEM simulations. The prediction and quantification of particle breakage were made possible by assuming a known distribution of single particle crushing strength and using a crushing criterion which depends on the maximum contact force acting on a particle. The effects of initial sample grading and relative density on the particle breakage evolution, however, were not considered in that study.

Based on the work of Marketos and Bolton (2007, 2009), a new statistical method which evaluates the probability of particle crushing by taking into account the statistics of both particle-scale stress distribution and characteristic fracture strength is presented in this paper. Theoretical formulation is further implemented in the DEM simulations of 1D compression of granular soils. The effectiveness and validity of the theoretical and numerical models are demonstrated by comparing the simulation results with the available experimental data.
2. Methodology

2.1. The fracture criterion of a single particle

In the soil mechanics literature, the fracture strength of a single sand particle was conventionally evaluated using the single particle crushing test, in which the particle was loaded within a platen compression apparatus without the confining support until the particle was broken (e.g., Lee, 1992; McDowell and Bolton, 1998; Nakata et al., 1999; Frossard et al., 2012). The variability of the particle fracture strength, which usually was taken from the peak load during the test, was found to be well fitted by Weibull’s distribution (Weibull, 1951). This finding formed the basis of the statistical analysis method which emerged recently for the study of crushable soils but still remained under-developed. Considering that particle crushing is actually an independent microscopic deformation mechanism from particle rearrangement of a loaded granular soil (Wang and Yan, 2011), the importance of Weibull’s theory lies in its crucial role in the establishment of a particle fracture criterion. What is missing to accomplish such a goal is then merely the information of the applied stress condition of a particle.

A major limitation of all the previous Weibull’s equations for particle crushing (e.g., McDowell and Bolton, 1998; Nakata et al., 1999; Frossard et al., 2012) is that they only considered the simple, uncon fined compression condition of a particle, which is rarely the case in a real loaded sample, where any particle has frequently more than two contacts with its neighboring particles. Therefore, the first contribution of this paper is the improvement of the existing Weibull’s equation for single particle fracture through the consideration of a random number of contact forces acting on a particle.

The above goal is conveniently achieved by computing the maximum tensile stress from the average stress tensor within a particle. Consider a two-dimensional particle with diameter $d$ subjected to an arbitrary number of contact forces, as shown in Fig. 1. Assuming the particle to be homogeneous, the average stress tensor $\sigma_{ij}$ within the particle can be written as (Tsoungui et al., 1999)

$$\sigma_{ij} = \frac{d}{2V_{pc}} \sum_{c=1}^{n_c} n_i^{(c)} f_j^{(c)},$$

where $n_c$ is the number of contacts of the particle; $n_i^{(c)}$ and $f_j^{(c)}$ are the $i$th component of the unit contact normal vector and the $j$th component of the contact force vector at $c$th contact, respectively; and $V_{pc}$ is the particle volume. The major and minor principal stresses acting on the particle and the angle of the major principal stress direction from the horizontal can then be calculated as

$$\sigma_{1,3} = \frac{\sigma_{11} + \sigma_{33}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{33}}{2}\right)^2 + \sigma_{13}\sigma_{31}},$$

$$\phi = \frac{\pi}{2} + \frac{1}{2} \arctan\left(\frac{2\sigma_{13}}{\sigma_{11} - \sigma_{33}}\right),$$

where $\sigma_1$ and $\sigma_3$ are the major and the minor principal stresses, respectively.

According to the modified "Brazilian" criterion (Tsoungui et al., 1999; Ben-Nun et al., 2010), the maximum tensile stress of a particle is perpendicular to the major principal stress direction and calculated as

$$\sigma_t = \frac{1}{4}(\sigma_1 - 3\sigma_3),$$

Note that positive or negative sign of $\sigma_t$ indicates that the particle is in tension or compression along the minor principal stress direction, respectively. For simplicity, it is assumed in this study that particle fracture is solely caused by the tensile stress and will occur along the plane of maximum tensile stress (i.e., major principal stress direction). Then, particle crushing probability $p_c$ used by Nakata et al. (1999) can be modified as

$$p_c = 1 - \exp \left[-\left(\frac{d}{d_0}\right)^2 \left(\frac{\sigma_t}{\sigma_0}\right)^m\right],$$

where $d_0$, $\sigma_0$, and $m$ are the reference diameter, the characteristic tensile stress, and the Weibull modulus of the particle, respectively; $\sigma_t$ is the maximum tensile stress calculated by Eq. (4), replacing the one originally estimated using the maximum axial load and particle diameter from the platen compression test. From the results of Nakata et al. (1999), the reference particle diameter $d_0$ was selected as 2.0 mm. Eq. (5) will be used later to construct the joint-probability particle crushing criterion.
2.2. The statistics of particle-scale stress distribution in oedometer samples

In the following paragraphs, knowledge of particle-scale statistical stress distributions in an oedometer sample is to be acquired from 2D DEM simulations of idealized samples composed of rigid spheres. Our specific goal is to examine the influences of initial sample grading and applied stress level on the particle-scale stress distribution, the former factor not being looked into by Marketos and Bolton (2007).

All the simulations were performed using PFC2D (Itasca, 2008) in this study. Each sample has a dimension of 15 mm (W) × 15 mm (H) and an initial void ratio of about 0.22 (Fig. 2(a)). Three initial sample gradings were used, with the maximum particle diameter \( d_{\text{max}} \) being fixed at 2 mm, and the mean particle diameter \( d_{50} \) and the uniformity coefficient \( C_u \) (i.e., \( C_u = d_{60}/d_{10} \)) being changed to achieve the different initial gradings, as shown in Fig. 2(b). During the simulation, the sample was confined by two fixed lateral boundaries and compressed in the vertical direction by moving the top and bottom boundaries towards each other at a constant rate of 0.2 mm/s. The samples were loaded to four target vertical stresses of 2, 4, 8 and 12 MPa sequentially, and after each target stress was reached, the sample was allowed to consolidate fully before the particle-scale stress data were extracted. The DEM parameters are listed in Table 1. Note that the particle density and inter-particle friction coefficient follow the previous studies of the authors (e.g., Wang et al., 2007; Wang and Gutierrez, 2010; Zhou et al., 2013). The high values of particle and wall stiffness of \( 1 \times 10^9 \) N/m and \( 1 \times 10^{10} \) N/m, respectively, were used to minimize particle overlaps and obtain more precise particle-scale stress distributions. The particle stiffness value was chosen based on an estimate of the elastic modulus of 3.48 GPa of quartz particles subjected to single particle crushing tests by Nakata et al. (1999) and Cavarretta et al. (2010).

Simulation data of the probability density functions (pdfs) of particle-scale stress are shown in Figs. 3 and 4. Herein, the tensile stress index \( f \) is proposed to depict the tensile stress condition of a particle and defined as follows:

\[
f = \frac{\sigma_t}{\langle \sigma_t \rangle}
\]

where \( \sigma_t \) is calculated from Eq. 4; \( \langle \sigma_t \rangle \) is the average value of \( \sigma_t \) of all the particles.

A typical example (i.e., \( C_u = 1.45, \sigma = 4 \) MPa) of the spatial distribution of the tensile stress index \( f \) within the sample is shown in Fig. 5, where no clear pattern can be identified. However, the data of pdfs shown in Figs. 3 and 4 display a distinct unified pattern. Specifically, Fig. 3(a) and (b) shows the pdfs of \( f \) of the same sample under different stress levels in the linear and semi-logarithmic scales, respectively; whereas Fig. 4(a) and (b) shows the pdfs of \( f \) of different samples under two selected stress levels in the two scales. It is clear that the pattern of the pdfs is almost independent of the applied stress level and initial sample grading. The data in the gray zone have positive \( f \) values indicating that the particles are under tension and the data in the white zone have negative \( f \) values indicating that the particles are under compression. In the current study, we are only interested in the gray zone because particle fracture is assumed to be caused only by tensile stress. The possibility of other particle breakage modes such as asperity

![Fig. 2](image-url)
yielding and surface abrasion induced by compressive stress is still open to debate and thus not considered in this study.

To show further whether the above pdfs of the local stress are affected by microscopic DEM parameters, we summarized in Fig. 6 the effects of particle contact stiffness and particle friction coefficient on the pdfs, where the former parameter varies from 0.5 to 4.0 GPa and the latter parameter varies from 0.3 to 0.7. As compared to Figs. 3 and 4, it is found that the variation of these two parameters has only very slight influence on the pdfs of the local stress index.

Given the unified pdfs from all the simulations, we propose the following best-fit function for the tensile stress index $f$:

$$g_i(f) = 0.46(f + 1.6)^{0.78}\exp(-1.98f)$$

(7)

This result reflects the intrinsic feature of the similar geometric packings developed within the rigid-particle samples to withstand external loads. It is assumed that Eq. (7) is valid within a much wider range of the applied stress and initial sample grading tested in the above simulations. Furthermore, a simple linear correlation was obtained between the average tensile stress $\langle \sigma_t \rangle$ of all the particles and the applied vertical stress $\sigma$ at the boundaries from all the simulations, as shown in Fig. 7. It can be simply fitted as

$$\langle \sigma_t \rangle = 0.38\sigma$$

(8)

With Eq. (6), the maximum tensile stress $\sigma_t$ of a random particle within the assembly can be written as

$$\sigma_t = 0.38af.$$  

(9)

2.3. The particle size distribution in tension zone

It is now useful to find out the particle size distribution (PSD) in the tension zone of different samples. Firstly, a diameter index $I_d$ is introduced to express different PSDs in a
uniform non-dimensional scale and defined as

\[ I_d = \frac{d - d_{min}}{d_{max} - d_{min}}, \]  

(10)

where \(d\) is the current particle diameter; \(d_{max}\) and \(d_{min}\) are the maximum and the minimum particle diameters of the whole assembly, respectively. It is easy to know that \(I_d\) varies from 0 to 1. Next, the pdfs of all the particles within the tension zone in terms of the diameter index \(I_d\) are obtained and shown in Fig. 8. Specifically, all the particles in the tension zone from each simulation were selected and the probability density distribution of these particles according to the \(I_d\) value of each particle was constructed. It can be found from the selected data under the applied stresses of 4 MPa and 8 MPa shown in Fig. 8 that, an approximately linear form of pdfs is obtained for all the samples, with smaller particles having higher chances of being in tension; and all the pdfs seem to intersect at \(I_d = 0.5\) and rotate clockwise with the increasing sample grading (i.e., \(C_u\) value). The latter result indicates that in a very well-graded sample, the probability of a particle being in tension strongly depends on its size and the coordination number is the main deciding factor behind this. Since the uniformity coefficient \(C_u\) is the only variable affecting the pdfs, the following best-fit function is proposed for the pdfs of \(I_d\):

\[ g_2(I_d) = -1.59(I_d - 0.5) \ln C_u + 1.0. \]  

(11)

According to Eq. (11), it can be inferred that for a uniform packing with \(C_u = 1.0\), all the particles have the same probability of being in tension, while for a perfect packing with very large \(C_u\) values, it is impossible for large particles to be in tension. This numerical result seems to support the experimental conjecture of Altuhafi and Coop (2011a, 2011b) that there would exist a perfect packing in which no remarkable particle crushing happens during the loading. An important conclusion of the above results, therefore, is that the sample grading does not affect the stress distribution of all the particles, but does dominate the size-dependent allocation of tensile stresses.

### 2.4. Joint-probability particle crushing criterion

With the establishment of particle crushing probability function and particle-scale statistical stress distribution functions, it is now time to construct the joint-probability particle crushing criterion. Before doing so, however, it is first necessary to examine whether the two statistical functions \(g_1(f)\) and \(g_2(I_d)\) in Eqs. (7) and (11) are inter-correlated or not. This is made by calculating the correlation coefficient \(\rho\) between \(f\) and \(I_d\) with the following equation (Ang and Tang, 2007):

\[ \rho = \frac{E[(f - \langle f \rangle)(I_d - \langle I_d \rangle)]}{SD_f \cdot SD_{I_d}}, \]  

(12)

where \(\langle f \rangle\) and \(\langle I_d \rangle\) are the average values of \(f\) and \(I_d\), respectively; \(SD_f\) and \(SD_{I_d}\) are the standard deviations of \(f\) and \(I_d\), respectively; \(E\) is the expected value.
and $I_d$, respectively; and $E[x]$ is the expected value of the statistical variable $x$.

Since we are only interested in the tension zone, Eq. (12) is only applied to all the particles in tension and the $\rho$ values calculated for all the simulations are listed in Table 2. It is found that all the $\rho$ values are very small, although there is a trend of slightly increasing $\rho$ value with increasing sample grading and applied stress. This result indicates that the two statistical variables $f$ and $I_d$ of any particle are largely independent of each other. Further proof of this statement is provided in Fig. 9, where as an example, all the data points ($f$, $I_d$) of the particles being in tension from the simulation with $C_w=1.45$ and $\sigma=8.0$ MPa are plotted. It is clear that the vast majority of the data points are randomly distributed in the whole area, lacking of any distinct correlation pattern.

Therefore, the joint pdf of a particle with a diameter index $I_d$ and a tensile stress index $f$ can be expressed as

$$G(f, I_d) = g_1(f)g_2(I_d).$$

Substituting Eqs. (7) and (11) into Eq. (13) yields

$$G(f, I_d) = 0.46(f+1.6)^{0.78}\exp(-1.98f)$$

$$\times[-1.59(I_d-0.5)\ln C_u+1.0], f \in [0, +\infty); I_d \in [0, 1].$$

The three-dimensional plot of $G(f, I_d)$ for the case with $C_w=1.45$ and $\sigma=8.0$ MPa is illustrated in Fig. 10. If we denote $A$ as the event of a particle having the tensile stress index $f$, and $B$ as the event of a particle having the diameter index $I_d$, then the joint probability of simultaneous events of $A$ and $B$ is

$$p(AB) = G(f, I_d)df\,dl_d$$

where $df$ and $dl_d$ are the infinitesimal increments of $f$ and $I_d$, respectively. Further denoting $C$ as the event of particle crushing, the conditional probability of event $C$ upon the joint events of $A$ and $B$ can then be expressed as

$$p(C|AB) = p(AB)p(C|AB).$$

---

**Table 2**

<table>
<thead>
<tr>
<th>Correlation</th>
<th>$2$ MPa</th>
<th>$4$ MPa</th>
<th>$8$ MPa</th>
<th>$12$ MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_w=1.24$</td>
<td>$-0.102$</td>
<td>$-0.118$</td>
<td>$-0.123$</td>
<td>$-0.143$</td>
</tr>
<tr>
<td>$C_w=1.45$</td>
<td>$-0.106$</td>
<td>$-0.099$</td>
<td>$-0.101$</td>
<td>$-0.108$</td>
</tr>
<tr>
<td>$C_w=2.15$</td>
<td>$-0.217$</td>
<td>$-0.221$</td>
<td>$-0.225$</td>
<td>$-0.235$</td>
</tr>
</tbody>
</table>
Substituting Eqs. (9) and (10) into Eq. (5), the second term of the right-hand side of Eq. (16) can be written as

\[ p(t|AB) = p_e, \]

\[ = 1 - \exp \left[ - \left( \frac{f_d (d_{max} - d_{min}) + d_{min}}{d_0} \right)^2 \left( \frac{0.38 \sigma_f}{\sigma_0} \right)^m \right]. \]

(17)

Then, substituting Eqs. (15) and (17) into Eq. (16) yields the joint-probability particle crushing criterion

\[ p = G(f; I_d) \left\{ 1 - \exp \left[ - \left( \frac{f_d (d_{max} - d_{min}) + d_{min}}{d_0} \right)^2 \left( \frac{0.38 \sigma_f}{\sigma_0} \right)^m \right] \right\} \, df \, dl_d, \]

(18)

The total crushing probability of a sample can then be found by integrating Eq. (18) as

\[ P_{crush} = \int_{f_0}^{\infty} \int_{I_{d0}}^{\infty} G(f; I_d) \left\{ 1 - \exp \left[ - \left( \frac{f_d (d_{max} - d_{min}) + d_{min}}{d_0} \right)^2 \left( \frac{0.38 \sigma_f}{\sigma_0} \right)^m \right] \right\} \, df \, dl_d, \]

(19)

where \( G(f; I_d) \) is given by Eq. (14).

An approximate numerical solution of Eq. (19) can be obtained from

\[ P_{crush} = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} G(f_i; I_{d_j}) \left\{ 1 - \exp \left[ - \left( \frac{f_d (d_{max} - d_{min}) + d_{min}}{d_0} \right)^2 \left( \frac{0.38 \sigma_f}{\sigma_0} \right)^m \right] \right\} \Delta f \, \Delta I_d, \]

(20)

where \( \Delta f \) and \( \Delta I_d \) represent the finite-sized increments of \( f \) and \( I_d \), and \( N_1 \) and \( N_2 \) are the numbers of increments of \( f \) and \( I_d \), respectively, as shown in Fig. 10.

2.5. DEM simulation of particle crushing in oedometer tests

To verify the previous theoretical development, we also conduct DEM simulations of particle crushing in oedometer tests. The sample preparation and testing procedure are the same as that used previously for rigid-particle samples except that now particle crushing is allowed during the loading process. Different from the theoretical approach, the event of particle breakage will have a deterministic nature for any given particle in DEM simulations. A particle will crush once its applied stress condition meets the crushing criterion given below (Nakata et al., 1999)

\[ \sigma_t \geq \sigma_c = \left( \frac{d}{d_0} \right)^{-2/m} \sigma_0. \]

(21)

The crushed particle will then be replaced by 8 sub-particles inscribed in the mother particle, as shown in Fig. 11. In our simulations, to speed up the computation, we set to make the breakage check for all the particles every 100 time steps. In view of the possible continuous breakage of a particle and its fragmented sub-particles and to avoid the excessively increasing computational load, we prescribed the minimum particle size to be 0.1 mm, below which a particle will not break further in our simulations. This is supported by experimental evidence (Altuhaifi and Coop, 2011a, 2011b) showing that silt particles (smaller than
0.1 mm in diameter) do not show significant breakage. In the current study, the particle-cluster technique was not used to generate irregular-shaped agglomerates and investigate the particle shape effects mainly due to the difficulty in evaluating the local stress existing in an irregular agglomerate subjected to an arbitrary number of contact forces.

3. Results of theoretical and numerical model predictions of particle crushing

Fig. 12 shows the theoretical predictions of the crushing probability evolution in oedometer tests using Eq. (20). The effects of initial sample grading (i.e., via $C_u$), characteristic particle tensile stress $\sigma_0$ and the Weibull modulus $m$ on the crushing evolution are examined. It is seen that $\sigma_0$ has overall much greater effects than $C_u$ and $m$ on the crushing evolution profiles by comparing Fig. 12(a) with (b) and (c); but a well-defined profile typically consists of three stages: (1) an initial flat stage without significant particle breakage, (2) a long middle stage with rapidly increasing particle breakage, and (3) a final stable stage with largely reduced or ceased particle breakage. Stages 1 and 3 become particularly distinctive when $\sigma_0$ takes a very high and low value, respectively (Fig. 12(b)). Interestingly in Fig. 12(c), a fully different pattern of the crushing evolution profiles illustrating the effect of $m$ is found. The difference is essentially featured with a diminishing stage 1 but overall lower crushing probabilities in stage 3 for a low $m$ value (e.g., $m=1$). The curves with different $m$ values intersect at an approximate vertical stress of 25 MPa, a turning point marking the change of the curvature. The crushing probability decreases and increases, respectively, before and after the turning point with the increasing $m$ value. It should be noted that the above crushing evolution is solely driven by the increasing applied vertical stress and the influence of evolving soil grading is not taken into account since only the initial grading is used in the calculations.

Fig. 13 compares the DEM predictions of crushing probability evolution with the theoretical predictions from the simulations with $C_u=1.45$, two $\sigma_0$ values of $\sigma_0=8$ MPa and $\sigma_0=16$ MPa, and two values of $m=1.0$ and $m=4.0$, respectively. It is found that for the cases with a fixed $m$ value of 4.0, a better agreement exists between the DEM and theoretical predictions in the high $\sigma_0$ case, and the DEM predicts a much higher rate of the increase of and thus magnitude of the crushing probability than the theoretical approach after the onset of significant particle crushing in the low $\sigma_0$ case (Fig. 13(a)). For the cases with a fixed $\sigma_0$ value of 8.0 MPa, the change of $m$ value is found not to change the pattern of DEM prediction significantly, and the deviations from the theoretical predictions are both larger than the case with the higher $\sigma_0$ value (i.e., $\sigma_0=16$ MPa) (Fig. 13(b)). The difference is mainly caused by the automatic inclusion of the soil grading change and the associated soil fabric evolution in the DEM simulation.

More information, however, could be provided by DEM simulations. Fig. 14 compares the DEM results of the specific volume vs. vertical stress relationship from a dense sample ($e_0=0.17$) and a loose sample ($e_0=0.22$) with the experimental results in the literature. Note that the values of $\sigma_0$ and $C_u$ in the simulations are selected as 8.0 MPa and 1.45,
respectively. The experimental data include one dimensional compression data of silica sand by Nakata et al. (2001) and Leighton Buzzard sand by Altuhaﬁ and Coop (2011a, 2011b). It is found in Fig. 14 that the two DEM samples generate a unique normal compression line (NCL) whose starting position and slope are very close to those of the two selected sands. This result veriﬁes the capability of the DEM model in reproducing the realistic normal compression behavior of crushable sands and more essentially the soundness of the current probability-based particle crushing simulation method in predicting the normal compression rate of real sands.

Finally, we examine the evolution of PSD of the oedometer sample, a typical example of which is shown in Fig. 15. It is seen that on a double-logarithmic scale, the PSD keeps rotating clockwise and getting closer to a linear distribution at larger stresses. In the literature, the topology of a naturally crushed granular material is normally quantiﬁed using the fractal theory (e.g., Steacy and Sammis, 1991; McDowell and Daniell, 2001), which states that the PSD of a two-dimensional granular soil subjected to a continuous fragmentation process satisﬁes the following power law:

\[
\frac{M(<d)}{M_T} \propto \left( \frac{d}{d_{\text{max}}} \right)^{2-D} \tag{22}
\]

where \(M(<d)\) is the mass of all particles smaller than diameter \(d\); \(d_{\text{max}}\) is the largest particle size; \(M_T\) is the total mass of all the particles; and \(D\) is the fractal dimension.

On a PSD plot, the fractal dimension \(D\) corresponds to the slope of the linear proﬁle of the PSD. For real granular soils,
the range of measured values of $D$ is between $2.5 \pm 0.1$ (Turcotte, 1986; McDowell and Daniell, 2001). In the 2D case, the range of $D$ given by a few theoretical analyses (e.g., Sammis et al., 1987; Steacy and Sammis, 1991) is between $1.6 \pm 0.1$. In the current study, an approximate $D$ value of 1.3 is obtained, which is very close to the fractal dimension of an Apollonian gasket with 25 generations (i.e., $D=1.305684$) (Manna and Hermann, 1991; Ben-Nun et al., 2010). The difference from the theoretically predicted values could be attributed to several factors including the prescribed minimum crushable particle size of 0.1 mm, the choice of Weibull's modulus $m$ and the assumed particle crushing mode, all of which could affect the evolution of the PSD.

4. Conclusion

The major thrust of this paper lies in the theoretical development of a probability-based method for the quantification and prediction of particle crushing in one-dimensional compression of sands, and its numerical implementation via DEM simulation. By recognizing particle crushing as an independent micromechanical deformation mechanism from particle rearrangement and the statistical nature of particle breakage events during the loading process of a sand, the idea presented in this paper opens a new avenue towards the rational quantification of particle crushing effects and the incorporation of such effects into the development of a new generation of micromechanism-based constitutive models of sands. The current work expanded the previous work of Marketos and Bolton (2007) by proposing a particle crushing criterion that takes into account the general stress condition of a particle with a random coordination number and the size-dependency of the particle crushing strength, the former being achieved through the use of the homogenized tensile stress within the particle while the latter through the employment of Weibull's equation of particle crushing strength. Furthermore, the possibility of continuous crushing of a particle and its crushed fragments has been included in the current analysis, which was a goal set by Marketos and Bolton (2007) for their next phase of research.

Although the information of statistical particle-scale stress distributions used in the formulation of the particle crushing criterion was derived from DEM simulations of rigid-particle samples that did not include particle crushing, the effectiveness of the proposed particle crushing criterion has been demonstrated through the compression between the results of theoretical and numerical predictions. The DEM simulations adopting such a particle crushing criterion have further produced realistic one-dimensional compression behavior agreeing with the experimental results of real sands, which verifies the effectiveness of the proposed approach in predicting particle crushing behavior as the key microscopic mechanism controlling the one-dimensional compression rate of sands.

Further improvement of the current approach will depend on the elucidation of the micromechanical processes of fabric evolution, strain localization development, etc., associated with the statistical occurrence of particle crushing within the sample. With the incorporation of such physical mechanisms, the statistical approach will become more robust and reliable. Another major development in the next phase of our research will be the utilization of X-ray CT technique for the identification of the linkage between the fracture mode and microstructure of a single sand particle as well as the incorporation of this information into the probability-based DEM simulation of irregular-shaped agglomerates using the particle-cluster technique.

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